ACTUARIAL MATHEMATICS

Module Codes: AST 2333

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Aims and objectives of the module

This module provides an introduction to mathematical techniques relevant to actuarial work, focusing on both life and non-life insurance. It covers applied mathematics for exploring and analyzing cash flows, understanding life insurance principles, and applying mathematical concepts to general insurance. By the end of the module, students should understand the mathematics behind managing cash flows and various types of insurance contracts. They should be able to apply this knowledge to solve problems in finance, life insurance, and non-life insurance, demonstrating proficiency in communication, analytical techniques, and practical skills.

Contents of the module

- Actuarial mathematics: Financial
- Actuarial mathematics: Life
- Actuarial mathematics: General

Contents of the module

- Calculating present and accumulated values for various streams of cash flows as a basis for future use in:
- Reserving, valuation, pricing, asset/liability management,
- investment income, capital budgeting, and valuing contingent cash flows.

To calculate present and accumulated values for various streams of cash flows for reserving purposes, we can use the concept of net present value (NPV) and accumulated value (AV). Assume an insurance company needs to reserve funds for a series of annual payments of $1,000 each, starting at the end of year 1 and continuing for 5 years. The company uses an annual interest rate of 5% for its calculations. Present Value (PV): The present value of these future cash flows represents the amount of money needed today to cover these future payments.

Accumulated Value (AV): The accumulated value represents the future value of these payments at a specific point in the future.

Present Value Calculation

$$PV = \sum_{t=1}^{n} \frac{C_t}{(1+r)^t}$$

- PV: Present Value
- C_t: Cash flow in year t
- r: Interest rate per period
- n: Number of periods

Given:

$$C_t = \$1,000 \text{ for each year from year 1 to year 5}$$

$$r = 5\% \text{ or } 0.05$$

$$n = 5 \text{ years}$$

Calculating the present value:

$$PV = \frac{1000}{(1+0.05)^1} + \frac{1000}{(1+0.05)^2} + \frac{1000}{(1+0.05)^3} + \frac{1000}{(1+0.05)^4} + \frac{1000}{(1+0.05)^5}$$

$$PV = \frac{1000}{1.05} + \frac{1000}{1.1025} + \frac{1000}{1.1576} + \frac{1000}{1.2155} + \frac{1000}{1.2763}$$

Accumulated Value Calculation

$$AV = 1000 \times (1 + 0.05)^4 + 1000 \times (1 + 0.05)^3 + 1000 \times (1 + 0.05)^2 + 1000 \times (1 + 0.05)^1 + 1000$$

$$AV = 1000 \times 1.2155 + 1000 \times 1.1576 + 1000 \times 1.1025 + 1000 \times 1.05 + 1000$$

$$AV = 1215.50 + 1157.60 + 1102.50 + 1050.00 + 1000.00 = 5525.60$$

Therefore, the present value of the cash flows is \$4,329.48, and the accumulated value at the end of year 5 is \$5,525.60. These values are important for reserving funds to meet future obligations.

Valuation Example

Consider a bond with a face value of \$1,000, a coupon rate of 5%, and 5 years to maturity. The annual market interest rate is 4%. Calculate the present value of the bond.
Using the formula for present value of a bond:

$$PV = \frac{C}{(1+r)^1} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \frac{C}{(1+r)^4} + \frac{C+M}{(1+r)^5}$$

Where:

- C = Annual coupon payment = \$1,000 * 5% = \$50
- r = Annual market interest rate = 4%
- M = Face value of the bond = \$1,000

Calculating the present value:

$$PV = \frac{50}{1.04^1} + \frac{50}{1.04^2} + \frac{50}{1.04^3} + \frac{50}{1.04^4} + \frac{1050}{1.04^5}$$

$$PV = 48.08 + 46.15 + 44.36 + 42.71 + 888.49$$

$$PV = 1069.79$$

Therefore, the present value of the bond is \$1,069.79

Consider an insurance company offering a life insurance policy that pays a death benefit of \$100,000. The probability of death in the next year for a 40-year-old individual is 0.001. If the company charges a premium of \$200 for the policy, is it actuarially fair?
The actuarially fair premium is calculated as the expected present value of the benefit payments, which is the death benefit multiplied by the probability of death:

$$\text{Expected benefit} = \$100,000 \times 0.001 = \$100$$

Since the premium charged is \$200, which is higher than the expected benefit of \$100, the premium is not actuarially fair.

Asset/Liability Management Example

An insurance company has \$10 million in assets that are invested in bonds with an average annual return of 6%. The company has \$8 million in liabilities that must be paid in one year. The company expects to receive \$1 million in premium payments over the next year. Calculate the company's surplus at the end of the year.

The surplus at the end of the year is calculated as the difference between assets and liabilities, plus any income from investments and premiums:

$$\text{Surplus} = \$10,000,000 \times 1.06 - \$8,000,000 + \$1,000,000 = \$10,600,000 - \$8,000,000 + \$1,000,000 = \$$$

Therefore, the company's surplus at the end of the year is \$3.6 million.

Investment Income

Formula: Investment Income = Initial Investment * Interest Rate

Numerical Example: An individual invests $10,000 in a savings account that earns 3% interest annually. Calculate the investment income after one year.

Investment Income = $10,000 * 0.03 = $300

Formula: NPV = $\sum \left[\frac{CF_t}{(1+r)^t}\right]$ - Initial Investment

Numerical Example: A company is considering two projects, A and B, with the following cash flows (in thousands of dollars) and a discount rate of 10%:

- Project A: -$100, $30, $50, $40, $20
- Project B: -$200, $70, $90, $60, $40

Calculate the NPV for each project and determine which project the company should choose.

Valuing Contingent Cash Flows

Formula: Expected Value = $\sum$(Probability * Cash Flow)
Numerical Example: An investor is considering investing in a project that has a 50% chance of generating $100,000 in cash flows and a 50% chance of generating $50,000 in cash flows. Calculate the expected value of the project.
Expected Value = (0.5 * $100,000) + (0.5 * $50,000) = $75,000

Actuarial Mathematics in Life Insurance

Actuarial mathematics in the context of life insurance involves various aspects such as data collection, calculation of exposure to risk, derivation of decrement rates, and monitoring of actual versus expected experience. Here's a brief overview of these concepts:

- Data Collection: Actuaries collect and analyze data related to mortality (death rates) and other relevant factors, such as age, gender, occupation, and lifestyle habits. This data is used to understand past experience and make predictions for the future.
- Exposure to Risk Actuaries calculate the exposure to risk for a group of insured individuals. This involves determining the number of individuals at risk of experiencing a certain event (such as death) during a specific period.
- Derivation of Crude Decrement Rates: Decrement rates are used to quantify the probability of an insured individual experiencing a specific event (such as death or disability) within a given period. Crude decrement rates are derived from historical data and are used as a starting point for more refined calculations.
- Monitoring Actual vs. Expected Experience: Actuaries regularly compare the actual experience of a group of insured individuals (e.g., actual mortality rates) with the expected experience based on actuarial calculations. This helps insurers assess the accuracy of their assumptions and pricing.
- Mortality Variation: Actuaries also study how mortality rates vary based on social, economic, and regional factors. This analysis helps insurers better understand and price risks associated with

Numerical Examples

Data Collection

An insurance company collects data on 1,000 policyholders and finds that, on average, 5 policyholders die each year.

Exposure to Risk

In a group of 10,000 policyholders, there are 500 policyholders aged 50-55. The exposure to the risk of death for this group over the next year is 500 person-years.

Derivation of Crude Decrement Rates

Based on historical data, the crude mortality rate for individuals aged 60-65 is 0.02 (i.e., 2% of individuals in this age group die each year).

Monitoring Actual vs. Expected Experience

An insurance company expected 100 deaths among its policyholders aged 70-75 this year based on actuarial calculations. However, the actual number of deaths in this age group was 120. This indicates that the actual experience exceeded the expected experience, suggesting that the mortality assumptions

Introduction to the theory and techniques of life insurance and pensions covers a wide range of topics related to the design, pricing, and management of life insurance and pension plans. Here's a brief overview of some key concepts:

Life Insurance: Life insurance provides financial protection to individuals and their families in case of premature death. It involves the payment of premiums to an insurance company, which then pays out a lump sum or periodic payments to the beneficiaries of the policyholder upon their death. The theory of life insurance includes the principles of risk pooling, mortality risk, and the calculation of premiums based on actuarial principles.

Pensions: Pensions are retirement plans that provide a regular income to individuals after they retire from work. There are various types of pension plans, including defined benefit plans, defined contribution plans, and hybrid plans. The theory of pensions involves considerations such as funding levels, investment strategies, and the calculation of pension benefits based on factors such as salary history and years of service.

Risk Management: Both life insurance and pensions involve managing various risks, including mortality risk, investment risk, and longevity risk. Techniques such as risk pooling, diversification, and asset-liability matching are used to manage these risks and ensure the financial stability of insurance companies and pension funds.

Regulation and Compliance: The insurance and pension industries are subject to regulation to protect

Endowments: Provide a lump sum payment upon maturity or death within a specified period.
Annuities: Provide a series of payments over a specified period or for life.
Assurances: Provide a lump sum payment upon death.

Consider different states a policyholder can be in (e.g., alive, disabled, deceased) and calculate benefits and premiums accordingly.

General Benefits and Premiums: Calculate benefits and premiums for various scenarios, taking into account factors such as age, gender, and health status.

Two Lives and More General Multi-Life Functions:

- **Joint Life Status**: Benefits are paid until the death of either of the two insured individuals.
- **Last Survivor Status**: Benefits are paid until the death of the last surviving insured individual.

Other Concepts in Life Insurance

Multiple Decrements Model: Considers competing risks (e.g., death, disability) and calculates benefits and premiums accordingly.
Disability Model: Calculates benefits for individuals who become disabled and are unable to work.
Level and Variable Payments:

- **Level Payments**: Remain constant over time.
- **Variable Payments**: May increase or decrease based on factors such as investment performance.

Increasing and Decreasing Assurances and Annuities: Policies where the benefit amount increases or decreases over time.
Discrete and Continuous Time Payments:

- **Discrete Payments**: Occur at specific intervals, such as monthly or annually.
- **Continuous Payments**: Occur continuously over time.

Conclusion

These concepts are essential in the design, pricing, and management of life insurance products. Actuaries use them to assess risk and determine appropriate premiums and benefits.

Endowments:

- Formula: $FV = PV \times (1 + r)^n$
- Numerical Example: If you invest \$10,000 at 5% interest annually for 10 years, the future value would be \$16,288.90.

Annuities:

- Formula: $PV = \frac{C}{r} \times (1 - (1 + r)^{-n})$
- Numerical Example: If you receive \$1,000 at the end of each year for 5 years at 6% interest, the present value would be \$16,100.45.

Assurances:

- Formula: The present value is simply the lump sum payment. For example, if a life insurance policy pays out \$100,000 upon death, the present value would be \$100,000.

Actuarial notation is used to represent various aspects of life contingencies:

- q_x: Probability of a life aged x dying within one year
- A_x: Present value of a life aged x
- D_x: Present value of a life aged x receiving a death benefit

- q_x: The probability of a life aged x dying within one year.
  - Example: q_{30} represents the probability of a 30-year-old dying within one year.
- A_x: The present value of a life aged x.
  - Example: A_{35} represents the present value of a 35-year-old.
- D_x: The present value of a life aged x receiving a death benefit.
  - Example: D_{40} represents the present value of a death benefit for a 40-year-old.
- nP_x: The probability that a life aged x will survive n more years.
  - Example: $10P_{50}$ represents the probability that a 50-year-old will survive 10 more years.
- nq_x: The probability of a life aged x dying within n years.
  - Example: $5q_{60}$ represents the probability of a 60-year-old dying within 5 years.
- nE_x: The expected present value of benefits payable at the end of each year for n years to a life aged x.
  - Example: $10E_{70}$ represents the expected present value of benefits payable for 10 years to a 70-year-old.
- nI_x: The present value of an n-year temporary life annuity immediate on a life aged x.
  - Example: $15I_{80}$ represents the present value of a 15-year temporary life annuity immediate on an 80-year-old.

q_x: Life Insurance Pricing

q_{30} represents the probability of a 30-year-old dying within one year, which is used to calculate the premium for a one-year term life insurance policy for a 30-year-old.

Types of Reserves

- Incurred But Not Reported (IBNR): Claims that have occurred but not yet reported.
- Case Reserves: Set aside for known claims that have been reported but not settled.
- Unearned Premium Reserves: Reserved portion of premium for future coverage.
- Claim Development Reserves: Account for changes in cost estimates over time.

Reserves Calculation

- Involves complex actuarial techniques.
- Depends on factors like insurance type, historical claims, and regulations.
- Actuaries use statistical models to estimate future claims and liabilities accurately.

A_x: Reserves Calculation example

A_{35} represents the present value of a 35-year-old, which is used to determine the reserve needed to cover future benefits for a 35-year-old with a life insurance policy.

D_x: Death Benefits

D_{40} represents the present value of a death benefit for a 40-year-old, which is the amount paid out by a life insurance policy if the insured dies at age 40.

nP_x: Survival Probabilities

$10P_{50}$ represents the probability that a 50-year-old will survive 10 more years, which is used to calculate the expected payouts for a 10-year life annuity starting at age 50.

$_nq_x$: Probabilities of Death

$_5q_{60}$ represents the probability of a 60-year-old dying within 5 years, which is used to calculate the expected payouts for a pension plan over a 5-year period for a 60-year-old.

nE_x: Expected Benefits

$10E_{70}$ represents the expected present value of benefits payable for 10 years to a 70-year-old, which is used to determine the expected payouts for a 10-year life insurance policy for a 70-year-old.

nI_x: Present Value of Annuities

$15I_{80}$ represents the present value of a 15-year temporary life annuity immediate on an 80-year-old, which is used to determine the value of an annuity that pays out for 15 years starting at age 80.

The present value of an annuity is the current value of a series of future payments, discounted at a certain rate of interest. It represents the amount that would need to be invested today to yield the same total value as the future payments.
The formula for the present value of an annuity is given by:

$$PV = P \times \frac{1 - (1 + r)^{-n}}{r}$$

where:

- PV is the present value of the annuity,
- P is the amount of each payment,
- r is the interest rate per period, and
- n is the number of periods.

This concept is commonly used in finance and insurance to calculate the value of pension plans, mortgages, and other financial instruments.

Expected Present Values of Standard Products

The expected present value of a standard life insurance product is the sum of the present values of all possible outcomes weighted by their respective probabilities.

Actuarial principles and techniques are used to determine appropriate premiums and reserves:

- Net premium method
- Gross premium method
- Use of actuarial tables

Principle of Equivalence

The principle of equivalence states that the present value of premiums should be equal to the present value of benefits, expenses, and profit margins.

Risk classification in actuarial mathematics is a crucial aspect of insurance pricing. It involves categorizing policyholders into different risk classes based on factors that are predictive of future claims. These factors typically include age, gender, health status, occupation, and lifestyle habits.
The goal of risk classification is to ensure that insurance premiums are fair and reflect the actual risk of each policyholder. By grouping individuals with similar risk profiles together, insurers can more accurately estimate the likelihood of claims and set appropriate premiums for each group.
There are several methods used in risk classification, including:

- **Underwriting:** Underwriters assess the risk of individual policyholders based on their application forms, medical exams, and other relevant information. They use this information to determine the appropriate premium for each policyholder.
- **Actuarial Models:** Actuaries use statistical models to analyze historical data and predict future claims. These models take into account various risk factors and help insurers set premiums that align with expected claims experience.
- **Experience Rating:** Insurers may adjust premiums based on their own claims experience with similar policyholders. If a particular group has a higher-than-expected number of claims, premiums for that group may be increased.
- **Community Rating:** In some cases, insurers use a community rating approach, where all policyholders are charged the same premium regardless of individual risk factors. This approach is

Underwriting

An insurance company categorizes applicants into different risk classes based on their medical history. For example, a 40-year-old non-smoker with no pre-existing conditions might be classified as low risk, while a 60-year-old smoker with a history of heart disease might be classified as high risk. The premiums for these individuals would be determined based on their risk class.

Actuaries use a statistical model to analyze the claims experience of a group of policyholders over the past five years. Based on this analysis, they predict that the group will experience an average of 100 claims per year over the next five years. Using this information, they calculate the premiums needed to cover these expected claims, taking into account factors such as inflation and investment returns.

Experience Rating

An insurer notices that a particular group of policyholders, aged 50-55, has had a higher-than-expected number of claims over the past year. To account for this higher risk, the insurer decides to increase the premiums for this group in the coming year.

Community Rating

In a community-rated health insurance plan, all policyholders pay the same premium regardless of their individual risk factors. For example, all policyholders in a particular region might pay $500 per month for coverage

Non-life insurance, also known as property and casualty insurance, provides protection against financial losses other than those covered by life insurance. It covers assets and liabilities subject to various risks.

Key Elements

- **Risk Transfer:** Allows transfer of financial risk to insurer.
- **Pooling of Risk:** Many contribute premiums to a pool.
- **Indemnity:** Aims to restore insured's financial position.
- **Diversification:** Insurers offer various products and operate in multiple regions.
- **Regulation:** Subject to government regulation to ensure solvency and obligations.
- **Types of Policies:** Includes property, liability, and health insurance.

Conclusion

Non-life insurance plays a crucial role in helping manage financial risks associated with unforeseen events, providing peace of mind and financial security.

Suppose a homeowner purchases property insurance with a coverage limit of $300,000. If their house is damaged by a covered event such as a fire, the insurance company agrees to cover the cost of repairs up to $300,000, transferring the financial risk of the damage to the insurer.

Pooling of Risk

Consider an auto insurance company with 10,000 policyholders, each paying an annual premium of \$1,000. If the company expects that 1% of policyholders will file a claim averaging \$10,000 each, the total expected claims amount to \$1,000,000. By pooling the premiums from all policyholders, the insurer can cover these claims even though only a small percentage of policyholders experience losses.

Indemnity

An individual purchases health insurance with a deductible of $500 and a coinsurance rate of 20%. If they incur medical expenses of $3,000, the insurer will pay $2,300 ($3,000 - $500 deductible = $2,500 x 20% coinsurance = $500), indemnifying the insured for the remaining $2,300.

An insurance company offers both property insurance and marine insurance. If a hurricane causes significant damage to properties in one region, the losses from property insurance claims may be offset by the profits from marine insurance policies in unaffected regions, demonstrating how diversification helps mitigate risk.

Regulation

Government regulations require insurance companies to maintain a minimum level of capital reserves based on their risk exposure. For example, an insurer with $100 million in liabilities may be required to maintain $10 million in reserves to ensure they can fulfill their obligations to policyholders.

Types of Insurance Policies

Actuarial mathematics involves the analysis and management of risks related to insurance policies. There are several types of insurance policies, including:

- **Life Insurance**: Provides a financial benefit to beneficiaries upon the death of the insured. Types include term life, whole life, and universal life insurance.
- **Health Insurance**: Covers medical expenses incurred by the insured, including doctor visits, hospital stays, and prescription medications.
- **Property Insurance**: Covers damage to the insured's property, such as their home or car, due to events like fire, theft, or natural disasters. Can include liability coverage.
- **Casualty Insurance**: Covers losses not related to property damage, such as liability for bodily injury or personal injury. Can include coverage for legal expenses.
- **Liability Insurance**: Covers claims made against the insured for bodily injury or property damage caused by their actions. Includes coverage for legal defense costs and damages awarded in lawsuits.

Actuaries use mathematical and statistical models to analyze the risks associated with these policies and determine appropriate premium rates and reserves to ensure that insurance companies can meet their obligations to policyholders.

Case study of insurance Policy

A business purchases liability insurance with coverage limits of $1 million per occurrence and $2 million aggregate. If a customer sues the business for $1.5 million due to an injury on the premises, the insurer will cover $1 million (per occurrence limit), with the business responsible for the remaining $500,000.

An actuary working in non-life insurance might analyze historical data on car accidents to estimate the likelihood of future accidents and the associated costs. Based on this analysis, the actuary could calculate the premiums that should be charged for car insurance policies to ensure that the insurance company is adequately compensated for the expected losses.

Suppose individual insurance claims follow a gamma distribution with shape parameter $k = 2$ and scale parameter $\theta = 1000$. If we want to model the aggregate claims over a year for a portfolio of 1000 policyholders, we can use the gamma distribution with shape parameter $n \times k = 1000 \times 2 = 2000$ and scale parameter $\theta \times n = 1000 \times 1000 = 1,000,000$.

Let's say the average number of claims per month for a certain type of insurance policy follows a Poisson distribution with mean $\lambda = 5$. The time between claims can then be modeled using an exponential distribution with rate parameter λ. So, the average time between claims would be $1/\lambda = 1/5 = 0.2$ months.

Pareto Distribution

Suppose the distribution of losses from catastrophic events follows a Pareto distribution with shape parameter $\alpha = 2$ and scale parameter $m = 100,000$. This means that 80

Normal Distribution

While not typically used for individual insurance losses, let's say we have data on the annual aggregate losses for a particular insurance portfolio that approximately follows a normal distribution with mean \$1,000,000 and standard deviation \$100,000.

Lognormal Distribution

For individual insurance losses, let's say we have data that shows a right-skewed distribution with a mean of \$10,000 and a standard deviation of \$5,000. We could model these losses using a lognormal distribution.

Suppose we want to model the time until a certain type of equipment fails. If the failure rate of the equipment increases over time (known as ”wear-out” failures), we might use a Weibull distribution with shape parameter $\alpha = 2$ and scale parameter $\lambda = 1000$ hours.

The Burr distribution is defined by its shape parameters c and k. For example, if $c = 2$ and $k = 3$, the distribution would have a shape similar to the gamma distribution but with heavier tails.

Risk Models Involving Frequency and Severity Distributions

Risk models in actuarial science often involve both frequency and severity distributions. The frequency distribution represents the number of claims or losses occurring within a given period, often modeled using a Poisson distribution or a negative binomial distribution. The severity distribution represents the amount of each claim or loss, often modeled using a gamma distribution, lognormal distribution, or Pareto distribution. By combining these two distributions, insurers can assess the overall risk and determine appropriate premiums and reserves.

Basic short-term insurance contracts provide coverage for a specific period, typically one year or less. These contracts include auto insurance, homeowners insurance, renters insurance, and other similar products. Insurers use actuarial models to price these contracts based on the expected frequency and severity of claims, as well as other factors such as policyholder demographics and location. By accurately pricing these contracts, insurers can ensure they have sufficient funds to cover potential claims while remaining competitive in the market.

In game theory, consider a scenario where two insurance companies are competing for customers in a market. Each company can choose to either lower its premiums to attract more customers or keep its premiums high to maximize profits. The outcome for each company depends on the strategy chosen by the other, leading to a strategic interaction where game theory can help analyze the optimal strategy for each company.

Optimum Strategies

Suppose an insurance company is considering whether to introduce a new type of insurance policy. The company can use optimization techniques to determine the premium that should be charged for the policy to maximize its profits, taking into account factors such as the expected number of policyholders and the potential costs of claims.

Decision Functions

A decision function could be used to determine whether an insurance claim should be approved or denied based on factors such as the type of coverage, the nature of the claim, and the policyholder's history. The decision function would map these factors to a decision on whether to pay the claim, helping the insurance company automate the claims process.

Risk Functions

A risk function could be used to assess the riskiness of investing in a particular type of insurance policy. For example, a risk function could quantify the potential losses associated with offering flood insurance in a high-risk area, helping the insurance company decide whether to offer this type of coverage and at what price.

Suppose an insurance company is considering whether to invest in a new technology to improve its claims processing efficiency. The minimax criterion would suggest that the company should evaluate the potential losses if the technology fails and compare them to the potential benefits if it succeeds. The company would then choose the option that minimizes the maximum possible loss.

Bayes Criterion

The Bayes criterion could be used to determine the optimal pricing strategy for an insurance policy based on the available information. For example, if new data becomes available on the likelihood of a certain type of claim, the insurance company could update its pricing strategy using the Bayes criterion to minimize the expected loss.

Claims Reserving and Run-off Triangles

Claims reserving is a crucial aspect of insurance, involving the estimation of the funds needed to settle all claims arising from policies issued by an insurance company. Run-off triangles are a common tool used in claims reserving, especially for long-tail insurance lines where claims may take many years to settle, such as liability insurance or workers' compensation.

Accident Year	Development Year 0	Development Year 1	Development Year 2	Development Year 3
2018	100,000	150,000	175,000	
2019	120,000	160,000		
2020	110,000			

In this example, the cell at the intersection of 2018 (Accident Year) and Development Year 1 shows that by the end of the first calendar year after the accident year 2018, $150,000 worth of claims have been reported for accidents that occurred in 2018.

Applications of Run-off Triangles

- Claims Development Patterns: Run-off triangles help in understanding the development patterns of claims over time, crucial for estimating ultimate claim amounts and setting reserves.
- Reserving Methods: Actuaries use run-off triangles along with various reserving methods to estimate the ultimate claim amounts and set reserves accurately.
- Management Reporting: Insurance companies use run-off triangles for management reporting to track the development of claims and assess the adequacy of reserves.
- Reinsurance Evaluation: Run-off triangles help in evaluating the effectiveness of reinsurance programs by tracking net claims after reinsurance.
- Regulatory Compliance: Insurance regulators often require companies to maintain run-off triangles for financial stability and compliance with regulations.

Bayesian statistics is a framework for probabilistic reasoning that provides a principled way to update beliefs about uncertain quantities in the light of new evidence. It revolves around three fundamental concepts:

- **Prior Distribution**: Represents the beliefs or uncertainty about a quantity of interest before observing any data. It encapsulates existing knowledge, expert opinions, or historical data.
- **Likelihood Function**: Quantifies the probability of observing the data given the parameter values. It represents the likelihood of the data under different assumptions about the parameter values.
- **Posterior Distribution**: Combines the prior distribution and the likelihood function to provide an updated estimate of the parameter values after observing the data. It represents the beliefs about the parameters after taking the data into account.

Bayes' Theorem

Bayes' theorem is the fundamental theorem of Bayesian statistics and provides a mathematical formula for updating beliefs:

$$P(\theta|x) = \frac{P(x|\theta) \times P(\theta)}{P(x)}$$

Where:

- $P(\theta|x)$ is the posterior distribution, representing the updated beliefs about the parameter θ given the observed data x.
- $P(x|\theta)$ is the likelihood function, representing the probability of observing the data x given the parameter θ.
- $P(\theta)$ is the prior distribution, representing the initial beliefs about the parameter θ before observing the data.
- $P(x)$ is the marginal likelihood or evidence, representing the probability of observing the data x averaged over all possible values of θ.

Credibility theory is a branch of actuarial science that deals with the analysis of insurance data to predict future claims or losses. It incorporates both historical data and external data to estimate risk and determine appropriate premium rates. Bayesian models are often used in credibility theory to account for uncertainties and update predictions based on new information.
Bayesian models in credibility theory use Bayesian statistics to estimate parameters, predict future claims, and assess risk. These models provide a flexible framework for incorporating prior knowledge, updating beliefs based on new data, and quantifying uncertainty. They are widely used in insurance and actuarial science to improve risk assessment and decision-making processes.

Reinsurance Treaties

Reinsurance treaties are agreements between insurance companies (the insurer) and reinsurance companies (the reinsurer) to share or transfer risks. The main types of reinsurance treaties include:

- **Proportional Reinsurance**: In proportional reinsurance, the reinsurer agrees to share a proportion of the premiums and losses with the insurer. The reinsurer's share remains constant regardless of the size of the loss.
- **Excess of Loss Reinsurance**: In excess of loss reinsurance, the reinsurer agrees to pay for losses that exceed a certain threshold (the retention) up to a specified limit. This type of reinsurance is used to protect against large losses.
- **Stop-Loss Reinsurance**: Stop-loss reinsurance is a type of excess of loss reinsurance where the reinsurer agrees to reimburse the insurer for losses exceeding a specified amount in total, rather than per occurrence. This provides protection against high aggregate losses.

Derivation of Distributions, Moments, and Moment Generating Functions

For each type of reinsurance treaty, the distribution of losses to the insurer and reinsurer can be derived based on the underlying distribution of the insured losses and the terms of the treaty. The moments (mean, variance, etc.) of these distributions can also be calculated. The moment generating function (MGF) can be used to derive moments and other properties of the distribution.

Ruin theory deals with the probability of an insurance company becoming insolvent due to claims exceeding its assets. It considers both continuous and discrete models:

- **Continuous Models**: In continuous ruin theory, the surplus process of the insurer is modeled as a continuous-time stochastic process, often using diffusion processes such as the Brownian motion. The probability of ruin is typically expressed as the probability that the surplus process falls below zero.
- **Discrete Models**: In discrete ruin theory, the surplus process is modeled as a discrete-time stochastic process, often using Markov chains or difference equations. The probability of ruin is calculated based on the probabilities of reaching certain states in the surplus process.

Ruin theory helps insurers assess their solvency risk and make informed decisions about reinsurance and risk management strategies.

The total amount of claims S in the collective risk model is often modeled as the sum of individual claim amounts $X_1, X_2, \ldots, X_n$ over a specified period:

$$S = X_1 + X_2 + \ldots + X_n$$

where n is the number of claims.

The compound distribution S can be described as a compound distribution, where the frequency of claims N follows a certain distribution (e.g., Poisson or negative binomial) and the severity of each claim X follows another distribution (e.g., gamma, lognormal, Pareto). The total amount of claims is then the convolution of these two distributions:

$$S = X_1 + X_2 + \ldots + X_N$$

where N is the number of claims and X_i are the individual claim amounts.

In the method of moments, the parameters of the frequency distribution and the severity distribution are estimated by equating sample moments to population moments. For example, for the gamma distribution, the sample mean and variance are equated to the population mean and variance to estimate the shape and scale parameters.

The moments of the compound distribution can be calculated to describe its central tendency and dispersion. The moment generating function (MGF) of the compound distribution can also be derived to facilitate the calculation of moments and other properties. Other properties of interest, such as the probability of ruin for an insurer or the expected total claims over a specified period, can be derived from the compound distribution.